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Dirichlet and Neumann problems to critical Emden–Fowler type equations

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Abstract We describe recent results on attainability of sharp constants in the Sobolev inequality, the Sobolev–Poincaré inequality, the Hardy–Sobolev inequality and related inequalities. This gives us the solvability of boundary value problems to critical Emden–Fowler equations.

Keywords Minimizers · Critical exponent · Emden–Fowler equation · Sobolev inequality · Sobolev–Poincaré inequality · Hardy–Sobolev inequality · p-Laplacian

Mathematics Subject Classification (2000) 49J10 · 35J20 · 35J60

1 Introduction

This article is an extended text of my lecture at the Workshop on Variational Analysis and PDEs in Erice, Sicily, July 2006. It contains recent results on attainability of sharp constants in critical Sobolev-type inequalities.

Let $n \geq 2$, and let Ω be a smooth compact n-dimensional Riemannian manifold (with or without boundary; in the sequel we omit the words "smooth", "compact" and "Riemannian"). For $1 < p < n$ we denote by $p^* = \frac{np}{n-p}$ the Sobolev conjugate to p, that is the critical exponent for the embedding $W_p^1(\Omega) \hookrightarrow L_q(\Omega)$.

Let us consider the classical Sobolev inequality:

$$\lambda_1(p, \Omega) = \inf_{v \in \overset{o}{W}{}_p^1(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p^*,\Omega}} > 0. \tag{I}$$

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Since the embedding operator $\overset{o}{W}{}^1_p(\Omega) \hookrightarrow L_{p^*}(\Omega)$ is noncompact, the problem of attainability of the norm of this operator (i.e. the problem of existence of an extremal function in the inequality (I)) is nontrivial. Similar problems were treated in many papers (see, e.g., [12,30], and further references therein). The answer heavily depends on the geometry of Ω.

It is well known that for Ω being a bounded domain in $\mathbb{R}^n$, the infimum in (I) is *not attained*. Moreover, this infimum does not depend on Ω and equals $\frac{1}{K(n,p)}$, where

$$K(n, p) \equiv \sup_{v \in \mathcal{C}_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\|v\|_{p^*,\mathbb{R}^n}}{\|\nabla v\|_{p,\mathbb{R}^n}} = \omega_{n-1}^{-\frac{1}{n}} \cdot k(n, p), \tag{1}$$

while

$$k(n, p) = n^{-\frac{1}{p}} \left(\frac{p-1}{n-p}\right)^{\frac{1}{p'}} \left(\mathcal{B}\left(\frac{n}{p}, \frac{n}{p'} + 1\right)\right)^{-\frac{1}{n}}$$

is the sharp constant in the Bliss inequality [11]. The second equality in (1) was obtained in [4,38]. Note that the supremum in (1) is attained only on radially symmetric functions with a *noncompact* support

$$w_\varepsilon(r) \equiv (\varepsilon + r^{p'})^{1-\frac{n}{p}}, \quad \varepsilon \in \mathbb{R}_+, \tag{2}$$

and on their translations and dilations.

On the other hand, see [30], the infimum in (I) for $\Omega \Subset \mathbb{R}^n$ can be attained, under certain additional assumptions, if one deals with the functions in $W^1_p(\Omega)$ which do not belong to $\overset{o}{W}{}^1_p(\Omega)$ but vanish on some part of $\partial\Omega$.

By standard argument it follows that under suitable normalization the minimizer in (I), if it exists, is a positive solution of the Dirichlet problem for the critical Emden–Fowler equation

$$-\Delta_p u = u^{p^*-1}; \qquad u\big|_{\partial\Omega} = 0 \tag{I$'$}$$

(here $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is p-Laplacian).

We give sufficient conditions of the existence of minimizers in (I) and similar inequalities. This gives us solvability of boundary value problems to corresponding Emden–Fowler type equations. Also we consider the qualitative properties of solutions. The article is mostly a survey; however, some results of §6 are new.

Our main tool is the concentration-compactness principle of Lions ([31]; see also [30]). It is used in various forms; for the problem (I) it can be reformulated as follows.

Proposition 1 ([15, Proposition 1.1]). *Let Ω be a manifold with boundary. Let the infimum of the problem (I) satisfy the inequality*

$$\lambda_1(p, \Omega) < \frac{1}{K(n, p)}.$$

Then the infimum is attained.

We underline that the attainability of infimum in critical case is extremely sensitive to weak perturbations of the functional. For example, Proposition 1 shows that a weak negative perturbation of the numerator in (I) can provide the existence of minimizer in "flat" bounded domains. This effect was discovered in [13] and was investigated for various problems in a number of papers, see, e.g. [19]. We do not consider perturbation terms which could help.

The text is organized as follows. In §§2–4 we describe the problems without weight. The problems with singular weights are treated in §§5–7. In §5 the weight singularity is situated in Ω or at $\partial\Omega$; in the last case we deal with the Neumann boundary value problem. In §§6–7 we consider the Dirichlet problem with the singularity at $\partial\Omega$.

Let us introduce some notations. $\mathbb{S}^n$ stands for the standard n-dimensional sphere (with unit scalar curvature). If $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ we suppose the center of sphere being at the origin. $\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ is the area of $\mathbb{S}^{n-1}$. $q' = \frac{q}{q-1}$ is the Hölder conjugate exponent to q. $\mathcal{B}$ is the Euler beta-function. We use letter C to denote various positive constants.

2 Critical Emden–Fowler equations without weight

Theorem 1 ([15, Theorem 3.1]). *Let $n \geq 2$. Suppose that Ω is an n-dimensional manifold with boundary, and the scalar curvature is positive at some point of Ω. Then for some $\beta > 0$ and for $1 < p < \frac{n+2}{3} + \beta$, the infimum in (I) is attained.*

For $\Omega \subset \mathbb{S}^n$ and $\beta = 0$, this statement was proved in [6] (for $p = 2$, see also [5] and [7]). On the other hand, it is shown in [6] that for any $1 < p < n$ the infimum in (I) is attained if $\Omega \subset \mathbb{S}^n$ is sufficiently large.

As the example of "flat" domain shows, the assumption on curvature of Ω cannot be removed. Also, Theorem 1 is sharp in another sence.

Theorem 2 ([15, Theorem 3.2]). *Let Ω be the spherical "hat"*

$$\Omega = \{(\theta, \phi_1, \dots, \phi_{n-1}) \in \mathbb{S}^n : \ 0 < \theta < \theta_*\}. \tag{3}$$

Given $\beta > 0$, there exists $\theta_*(\beta) > 0$ such that for $p \geq \frac{n+2}{3} + \beta$ the infimum in (I) on Ω is not attained. In particular, the infimum in (I) on hemisphere ($\theta_* = \pi/2$) is not attained for $p \geq \frac{n+1}{2}$. If $\beta \to 0$ then $\theta^*(\beta) \to 0$.

For $p > \frac{n+1}{2}$ and $\theta_* \leq \pi/2$, this statement was proved in [6]. In [8], the explicit upper bound for $\theta_*(\beta)$ was found. This upper bound is sharp for $\theta_* = \pi/2$ and for limits $\theta_* \to 0$, $\theta_* \to \pi$. Authors of [8] (as well as the author of survey) believe that it is always sharp.

If we replace $\overset{o}{W}{}^1_p(\Omega)$ by $W^1_p(\Omega)$ then the quotient (I) vanishes on constants. There are some natural ways to avoid this degeneration. The simplest one is to add a weak term in the numerator. We arrive at the critical embedding theorem

$$\lambda_2(p, \Omega) = \inf_{v \in W^1_p(\Omega) \setminus \{0\}} \frac{\|v\|_{W^1_p(\Omega)}}{\|v\|_{p^*,\Omega}} > 0 \tag{II}$$

(the norm of the numerator is defined as $\|v\|^p_{W^1_p(\Omega)} = \|\nabla v\|^p_{p,\Omega} + \|v\|^p_{p,\Omega}$).

Note that under suitable normalization the minimizer in (II), if it exists, is a positive solution of the Neumann problem

$$-\Delta_p u + u^{p-1} = u^{p^*-1}; \qquad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0. \tag{II$'$}$$

The problem (II') has a trivial solution $u \equiv 1$; however, it is shown in [33, Proposition 1.3] that for $p > 2$ the minimizer of (II) cannot be a constant.

Theorem 3 ([15, Theorem 4.1]). *Let $n \geq 5$. Suppose that Ω is an n-dimensional manifold without boundary, and the scalar curvature is positive at some point of Ω. Then for some $\beta > 0$ and for $2 < p < \frac{n+2}{3} + \beta$, the infimum in (II) is attained.*

For $2 < p < \sqrt{n}$ the equivalent statement was proved in [17].

Now let us define $\Omega_\varkappa$ as a "dilation" of Ω with the metric $g(\Omega_\varkappa) = \varkappa^2 \cdot g(\Omega)$. Since the quotient (II) is not homogeneous with respect to dilations, the attainability of infimum, generally speaking, depends on $\varkappa$.

Let $n \geq 2$, and let Ω be an arbitrary n-dimensional manifold without boundary or with a strictly Lipschitz boundary. Then for any $1 < p < n$ the infimum in (II) is attained on $\Omega_\varkappa$ if $\varkappa$ is sufficiently small ([33, Theorem 1.1]).

On the other hand, the so-called **optimal Sobolev inequality**

$$\|v\|^p_{p^*,\Omega} \leq K^p(n,p) \cdot \|\nabla v\|^p_{p,\Omega} + C(p,\Omega) \cdot \|v\|^p_{p,\Omega}, \qquad v \in W^1_p(\Omega), \tag{4}$$

holds true with some $C(p,\Omega) > 0$:

- on $\Omega = \mathbb{S}^n$ for $1 < p < 2$ ([4, Theorem 8]);
- on an arbitrary manifold Ω without boundary for $n \geq 3$ and $p = 2$ ([25]);
- on the flat torus $\Omega = \mathbb{T}^n$ and on the hyperbolic manifold without boundary $\Omega = \mathbb{H}^n$ for any $1 < p < n$ ([17]).

It is easy to see that (4) implies the non-attainability of infimum in (II) on $\Omega_\varkappa$ for sufficiently large $\varkappa$. This implies that, in general, the assumption on curvature of Ω in Theorem 3 cannot be removed, and the lower bound on interval for p cannot be reduced.

If Ω is a manifold with (smooth) boundary then the mean curvature of $\partial\Omega$ (with respect to the inner normal) plays the role of scalar curvature.

Theorem 4 ([15, Theorem 7.1]). *Let $n \geq 2$. Suppose that Ω is an n-dimensional manifold with smooth boundary, and $\partial\Omega$ contains a point with the positive mean curvature. Then for some $\beta > 0$ and for $1 < p < \frac{n+1}{2} + \beta$, the infimum in (II) is attained. In particular, it is the case if $\Omega \subset \mathbb{R}^n$ is arbitrary bounded domain with $\partial\Omega \in \mathcal{C}^2$.*

For $\Omega \subset \mathbb{R}^n$, $n \geq 3$ and $p = 2$ this result was obtained in [1] and [40].

In general, the assumption of the smoothness of $\partial\Omega$ is essential.

Theorem 5 ([15, Theorem 7.2]). *Let $n \geq 2$. Suppose Ω is a polyhedron in $\mathbb{R}^n$, and $1 < p < n$. Then the infimum in (II) is not attained on $\Omega_\varkappa$ for sufficiently large $\varkappa$.*

For $n \geq 3$ and $p = 2$, for Ω being a rectangular parallelepiped, this statement was proved in [29, Theorem 1.2], using delicate properties of the solution of the corresponding boundary value problem.

3 The problems with sign-changing solutions

In this section we consider two more methods to avoid degeneration on constants. In contrast to (I) and (II), the solutions of corresponding problems, if they exist, change sign in Ω.

The first one is to subtract the mean value in the denominator of (I). We arrive at the Sobolev–Poincaré inequality

$$\lambda_3(p,\Omega) = \inf_{v \in W^1_p(\Omega)\setminus\{c\}} \frac{\|\nabla v\|_{p,\Omega}}{\|v - \overline{v}\|_{p^*,\Omega}} > 0 \tag{III}$$

(here we use the notation $\overline{v} = |\Omega|^{-1} \int_\Omega v$).

Theorem 6 ([15, Theorem 5.1]). *Let $n \geq 3$. Suppose that Ω is an n-dimensional manifold without boundary, and the scalar curvature is positive at some point of Ω. Then for some $\beta > 0$ and for $1 < p < \frac{n+2}{3} + \beta$, the infimum in (III) is attained.*

For $n \geq 4$ and $\frac{n}{n-1} < p < \frac{1}{4}(1+\sqrt{1+8n})$ the statement of Theorem 6 was proved in [41, Theorem 1.2]. In addition, [41, Theorem 1.1] claims the attainability of infimum in (III) for $n \geq 2$ and $1 < p < \frac{1}{4}(1+\sqrt{1+8n})$ in the case $\Omega = \mathbb{S}^n$. However, the proof of this theorem has a gap.

Corresponding result for the manifolds with boundary reads as follows.

Theorem 7 ([15, Theorem 7.3]). *Let $n \geq 2$. Suppose that Ω is an n-dimensional manifold with smooth boundary, and $\partial\Omega$ contains a point with the positive mean curvature (in particular, it is the case if $\Omega \subset \mathbb{R}^n$ is arbitrary bounded domain with $\partial\Omega \in \mathcal{C}^2$). Then for some $\beta > 0$ and for $1 < p < \frac{n+1}{2} + \beta$, the infimum in (III) is attained.*

The second method is to subtract in the denominator of (I) the best approximation constant instead of the mean value. We arrive at the problem

$$\lambda_4(p,\Omega) = \inf_{v \in W_p^1(\Omega)\setminus\{c\}} \sup_{\alpha \in \mathbb{R}} \frac{\|\nabla v\|_{p,\Omega}}{\|v-\alpha\|_{p^*,\Omega}} > 0. \tag{IV}$$

Theorem 8 ([15, Theorem 6.1]). *Let $n \geq 3$. Suppose that Ω is an n-dimensional manifold without boundary, and the scalar curvature is positive at some point of Ω. Then for*

$$1 < p < \max\{p_1, p_2\} \tag{5}$$

the infimum in (IV) is attained, where

$$p_1 = 2n+1-\sqrt{3n^2+2n+1}, \qquad p_2 = \frac{n^2+6n+2+\sqrt{n^4+12n^3-8n+4}}{2(5n+4)}.$$

Certainly, the condition (5) in Theorem 8 is unsatisfactory, and we are quite sure that it can be weakened. However, we claim that some upper bound for p is necessary.

Theorem 9 ([15, Theorem 6.2]). *Let $n \geq 2$. Then for $p \geq \frac{n+1}{2}$ the infimum in (IV) on $\Omega = \mathbb{S}^n$ is not attained.*

In particular, this theorem disproves the statement of [41, Theorem 1.3].

Theorem 9 allows us to study the symmetry breaking of the extremal in the embedding theorem on the sphere

$$\widetilde{\lambda}_4(n,p,q) = \inf_{v \in W_p^1(\mathbb{S}^n)\setminus\{c\}} \sup_{\alpha \in \mathbb{R}} \frac{\|\nabla v\|_{p,\mathbb{S}^n}}{\|v-\alpha\|_{q,\mathbb{S}^n}} > 0 \tag{IVa}$$

for subcritical q. Note that the maximum with respect to α in (IVa) is attained whenever the relation

$$\int_{\mathbb{S}^n} |u|^{q-2}u\, dV_g = 0, \tag{6}$$

holds true with $u = v - \alpha$.

Theorem 10 ([15, Theorem 6.3]). **1**. *Let $n \geq 2$, $1 < p < \infty$, $q = p$. Then the extremal function of (IVa), satisfying (6), is antisymmetric with respect to θ, that is $u(\theta) = -u(\pi-\theta)$ for $0 < \theta < \pi/2$.*

2. *Let $n \geq 2$, $\frac{n+1}{2} \leq p < n$. Then there exists $\beta > 0$ such that for $q \in (p^*-\beta, p^*)$ the extremal function of (IVa), satisfying (6), is not antisymmetric with respect to θ.*

We also formulate the corresponding result for the manifolds with boundary.

Theorem 11 ([15, Theorem 7.4]). *Let $n \geq 2$. Suppose that Ω is an n-dimensional manifold with smooth boundary, and $\partial\Omega$ contains a point with the positive mean curvature (in particular, it is the case if $\Omega \subset \mathbb{R}^n$ is arbitrary bounded domain with $\partial\Omega \in \mathcal{C}^2$). Then the infimum in (IV) is attained for*

$$1 < p < \max\{\widetilde{p}_1, \widetilde{p}_2\},$$

where

$$\widetilde{p}_1 = \frac{3n+1-\sqrt{5n^2+2n+1}}{2}, \qquad \widetilde{p}_2 = \frac{n^2+3n+1+\sqrt{n^4+6n^3-n^2-2n+1}}{2(3n+2)}.$$

Remark 1 The exponents p_1, p_2, $\widetilde{p}_1$ and $\widetilde{p}_2$ are monotone functions of n with linear growth for large n. Moreover, these exponents satisfy

$$\begin{array}{llll} 1 < p_1 < p_2 & \text{for} \quad 3 \leq n \leq 5; & 1 \leq \widetilde{p}_1 < \widetilde{p}_2 & \text{for} \quad 2 \leq n \leq 8; \\ p_1 = p_2 = 2 & \text{for} \quad n = 6; & 1 < \widetilde{p}_2 < \widetilde{p}_1 & \text{for} \quad n \geq 9. \\ 2 < p_2 < p_1 & \text{for} \quad n \geq 7; & & \end{array}$$

Note that under suitable normalization the minimizers in (III) and (IV) are sign-changing solutions of the Neumann problems for equations $-\Delta_p u = u^{p^*-1} - C$ and $-\Delta_p u = u^{p^*-1}$, respectively.

4 More general equations in $\Omega \subset \mathbb{R}^n$

When $\Omega \subset \mathbb{R}^n$ one can consider the quotients (I)–(IV) with a more general definition of $\|\nabla u\|_{p,\Omega}$. Namely, let us consider an arbitrary norm $\mathcal{M}(x)$ in $\mathbb{R}^n$. Assume, for simplicity, that $\mathcal{M} \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\})$, and the function $\mathcal{M}$ is strictly convex in non-radial directions. Denote by $\mathcal{M}_o(x)$ the conjugate norm. In particular, if

$$\mathcal{M}(x) = |x|_q \equiv \left(\sum_{k=1}^n |x_k|^q\right)^{\frac{1}{q}}, \quad 1 < q < \infty,$$

then $\mathcal{M}_o(x) = |x|_{q'}$.

Let $\omega_{n-1,\mathcal{M}}$ stand for the area of a unit sphere $\{x \in \mathbb{R}^n : \mathcal{M}(x) = 1\}$. In particular, for $\mathcal{M}(x) = |x|_q$ we have $\omega_{n-1,\mathcal{M}} = \omega_{n-1,q} = \frac{q[2\Gamma(\frac{1}{q}+1)]^n}{\Gamma(\frac{n}{q})}$; it is evident that $\omega_{n-1,2} = \omega_{n-1}$.

We introduce an equivalent seminorm $\|\nabla v\|_{p,\mathcal{M},\Omega}$ in $W_p^1(\Omega)$ by

$$\|\nabla v\|_{p,\mathcal{M},\Omega}^p = \int\limits_\Omega \mathcal{M}^p(\nabla v)\, dx.$$

Let us replace $\|\nabla v\|_{p,\Omega}$ in the numerators of quotients (I)–(IV) by $\|\nabla v\|_{p,\mathcal{M},\Omega}$. Corresponding problems are denoted by $(I_\mathcal{M})$–$(IV_\mathcal{M})$.

It is proved in [3] by using a convex symmetrization, that

$$K(n,p,\mathcal{M}) \equiv \sup_{v \in \mathcal{C}_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\|v\|_{p^*,\mathbb{R}^n}}{\|\nabla v\|_{p,\mathcal{M},\mathbb{R}^n}} = \omega_{n-1,\mathcal{M}}^{-\frac{1}{n}} \cdot k(n,p). \tag{7}$$

The supremum in (7) is attained only on the functions $w_\varepsilon(r)$, defined by (2), with $r = \mathcal{M}_o(x)$. Note that the same statement was obtained in [14] by using the mass transportation approach (the generalized Monge–Kantorovich problem).

This statement implies that for Ω being a bounded domain in $\mathbb{R}^n$, the infimum in $(I_{\mathcal{M}})$ is not attained, it does not depend on Ω and equals $\frac{1}{K(n,p,\mathcal{M})}$.

Theorem 12 ([15, Theorem 9.1]). *The statement of Theorem 4 is valid for the problem $(II_{\mathcal{M}})$; the statement of Theorem 7 is valid for the problem $(III_{\mathcal{M}})$; the statement of Theorem 11 is valid for the problem $(IV_{\mathcal{M}})$.*

Note that under suitable normalization the minimizers in $(II_{\mathcal{M}})$–$(IV_{\mathcal{M}})$ are solutions of the Neumann problems

$$\left.\begin{array}{lll} -\Delta_{p,\mathcal{M}} u = u^{p^*-1} - u^{p-1} & \text{for} & (II_{\mathcal{M}}), \\ -\Delta_{p,\mathcal{M}} u = u^{p^*-1} - C & \text{for} & (III_{\mathcal{M}}), \\ -\Delta_{p,\mathcal{M}} u = u^{p^*-1} & \text{for} & (IV_{\mathcal{M}}), \end{array}\right\}; \quad \langle \mathbf{n} \cdot \nabla \mathcal{M}(\xi)\rangle \Big|_{\substack{\xi=\nabla u\\ x\in\partial\Omega}} = 0.$$

Here

$$\Delta_{p,\mathcal{M}} u = \operatorname{div}\Big(\mathcal{M}^{p-1}(\xi)\nabla\mathcal{M}(\xi)\Big|_{\xi=\nabla u}\Big)$$

is the generalized p-Laplacian, generated by $\mathcal{M}$. In particular, for $\mathcal{M}(x) = |x|_q$ we have

$$\Delta_{p,\mathcal{M}} u = \Delta_{p,q} u \equiv \sum_{j=1}^{n} \left(|\nabla u|_q^{p-q} |u_{x_j}|^{q-2} u_{x_j}\right)_{x_j}.$$

Notice that $\Delta_{p,2}$ is the conventional p-Laplacian, while $\Delta_{p,p}$ is the so-called pseudo-p-Laplacian (see, e.g. [10,26]). The properties of the solutions of the Dirichlet problem for the equations with the operator $\Delta_{p,\mathcal{M}}$ of a general form and a subcritical right-hand side, were studied in [9].

Theorem 13 ([15, Theorem 9.3]). *Let $n \geq 2$. Suppose that Ω is the flat torus $\mathbb{T}^n$ or a polyhedron in $\mathbb{R}^n$, $1 < p < n$. Then the infimum in $(II_{\mathcal{M}})$ is not attained on $\Omega_\varkappa$ for sufficiently large $\varkappa$.*

5 The equations with singular weight. Interior singularities. The Neumann problem with boundary singularity

Let $x_0 \in \overline{\Omega}$. Consider a weighted Lebesgue space $L_{q,\sigma}(\Omega)$ with the norm

$$\|v\|_{q,\sigma,\Omega} = \|r^{\sigma-1} v\|_{q,\Omega},$$

where $r = \operatorname{dist}(x, x_0)$. For $\Omega \subset \mathbb{R}^n$ we can assume, without loss of generality, $x_0 = 0$.

For $1 < p < \infty$ and $0 \leq \sigma \leq \min\{1, \frac{n}{p}\}$ we denote by $p^*_\sigma = \frac{np}{n-\sigma p}$ the critical exponent for the embedding $W^1_p(\Omega) \hookrightarrow L_{q,\sigma}(\Omega)$. In this section we deal with the case $p < n$.

For the corresponding embedding theorem in $\mathbb{R}^n$

$$K(n, p, \sigma) \equiv \sup_{v \in \mathcal{C}_0^\infty(\mathbb{R}^n)\setminus\{0\}} \frac{\|v\|_{p^*_\sigma,\sigma,\mathbb{R}^n}}{\|\nabla v\|_{p,\mathbb{R}^n}} < \infty, \tag{8}$$

the case $\sigma = 1$ leads to the Sobolev inequality, while the case $\sigma = 0$ provides the Hardy inequality. Therefore, (8) is often referred as the Hardy–Sobolev inequality.

It is proved in [19], see also [23], that for arbitrary $1 < p < n$ and $0 < \sigma < 1$, the supremum in (8) is attained only on radial functions with a noncompact support

$$w_{\varepsilon,\sigma}(r) \equiv \left(\varepsilon + r^{p' \frac{\sigma p_\sigma^*}{p^*}}\right)^{-\frac{n}{\sigma p_\sigma^*}}, \quad \varepsilon \in \mathbb{R}_+.$$

The case $p = 2, n \geq 3$, was treated earlier in [28]. The value of supremum in (8) equals (see [15, §8])

$$K(n, p, \sigma) = \omega_{n-1}^{-\frac{\sigma}{n}} \left(\frac{\sigma}{n}\right)^{\frac{1}{p}} \left(\frac{p-1}{n-p}\right)^{\frac{1}{p'}} \left(\frac{p^*}{\sigma p_\sigma^*}\right)^{\frac{1}{p_\sigma^*}} \left(\mathcal{B}\left(\frac{n}{\sigma p}, \frac{n}{\sigma p'} + 1\right)\right)^{-\frac{\sigma}{n}}.$$

Let us replace the norm $\|\cdot\|_{p^*,\Omega}$ in the denominators of quotients (I)–(IV) by the norm $\|\cdot\|_{p_\sigma^*,\sigma,\Omega}$ and denote these problems by (I_σ)–(IV_σ).

The statements on attainability of infima in the problems (I)–(III) are generalized for problems (I_σ)–(III_σ).

Theorem 14 ([15, Theorem 8.1]). *Let $n \geq 2$. Suppose that Ω is an n-dimensional manifold with boundary. Let $x_0 \notin \partial\Omega$ be a point with the positive scalar curvature. Then, for a given $0 < \sigma < 1$, there exists $\beta > 0$ such that for $1 < p < \frac{n+2}{3} + \beta$ the infimum in (I_σ) is attained.*

Let $n \geq 5$. Suppose that Ω is an n-dimensional manifold without boundary. Let x_0 be a point with the positive scalar curvature. Then, for a given $0 < \sigma < 1$, there exists $\beta > 0$ such that for $2 < p < \frac{n+2}{3} + \beta$ the infimum in (II_σ) is attained.

Let $n \geq 3$. Suppose that Ω is an n-dimensional manifold without boundary. Let x_0 be a point with the positive scalar curvature. Then, for a given $0 < \sigma < 1$, there exists $\beta > 0$ such that for $1 < p < \frac{n+2}{3} + \beta$ the infimum in (III_σ) is attained.

Theorem 15 *Given $\beta > 0$, there exists $\theta_* > 0$ such that if Ω is the spherical "hat"* (3) *and $x_0 \in \Omega$ then for $p \geq \frac{n+2}{3} + \beta$ and for any $0 < \sigma < 1$ the infimum in (I_σ) is not attained.*

Proof By spherical symmetrization we reduce the general location of x_0 to the case where x_0 is the pole $\theta = 0$. In this case the statement is proved in [15, Theorem 8.2]. □

Theorem 16 ([15, Theorem 8.3]). *Let $n \geq 2$. Suppose that Ω is an n-dimensional manifold with a strictly Lipschitz boundary. Let $\partial\Omega \in \mathcal{C}^2$ in a neighborhood of the point $x_0 \in \partial\Omega$, and let $H(x_0) > 0$. Then, for a given $0 < \sigma < 1$, there exists $\beta > 0$ such that for $1 < p < \frac{n+1}{2} + \beta$ the infimum in (II_σ) is attained.*

Under the same assumptions, for a given $0 < \sigma < 1$, there exists $\beta > 0$ such that for $1 < p < \frac{n+1}{2} + \beta$ the infimum in (III_σ) is attained.

The analogs of Theorems 8 and 11 for the problem (IV_σ) can be proved by the same method. However, for $\sigma \neq 1$, the restrictions imposed on p turn out to be even more "wild". The analog of Theorem 12 for $0 < \sigma < 1$ is also obtained in [15].

In [20, §5] the problem (II_σ) was considered for $\Omega \subset \mathbb{R}^n, n \geq 3, x_0 \in \partial\Omega$ and $p = 2$. The last theorem of this paper (Theorem 5.4) claims, without proof, the attainability of infimum in (II_σ) for arbitrary $1 < p < n$. We conjecture, that this theorem is false without additional assumption $p \leq \frac{n+1}{2}$; under this assumption, it follows from the first part of our Theorem 16.

6 The Dirichlet problem with boundary singularity in cones and in perturbed cones

In this section we consider $\Omega \subset \mathbb{R}^n$ and arbitrary $1 < p < \infty$. Denote by $\dot{W}_p^1(\Omega)$ the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\nabla v\|_{p,\Omega}$. For bounded Ω it is obvious that $\dot{W}_p^1(\Omega) = \overset{o}{W}{}_p^1(\Omega)$.

Let us consider the Hardy–Sobolev inequality in Ω

$$\lambda_1(p,\sigma,\Omega) = \inf_{v \in \dot{W}_p^1(\Omega)\setminus\{0\}} \frac{\|\nabla v\|_{p,\Omega}}{\|v\|_{p_\sigma^*,\sigma,\Omega}} > 0. \tag{Iσ}$$

For $p < n$ and $0 \in \Omega$ it was discussed in §5.

Proposition 2 ([32, Propositions 2.1 and 2.2]). *Let $n < p < \infty$, and let $\Omega \subset \mathbb{R}^n \setminus \{0\}$. Then for any $0 \le \sigma \le \frac{n}{p}$ the inequality (I_σ) holds true.*

Let $p = n$, and let $\Omega \subset \mathbb{R}^n \setminus \ell$ where ℓ is a ray beginning at the origin. Then for any $0 \le \sigma < 1$ the inequality (I_σ) holds true.

Note that under suitable normalization the minimizer in (I_σ), if it exists, is a positive solution of the Dirichlet problem

$$-\Delta_p u = \frac{u^{p_\sigma^*-1}}{r^{(1-\sigma)p_\sigma^*}} \quad \text{in } \Omega; \qquad u\big|_{\partial\Omega} = 0. \tag{9}$$

It is worth to note that for $p = n$ the exponent in the denominator of (9) does not depend on σ and equals n.

Remark 2 Actually, for $p = n$ the assumption on Ω in Proposition 2 can be considerably weakened. However, for Ω being a cone, this assumption is sharp.

Theorem 17 ([32, Theorem 3.1]). *Let $1 < p < \infty$, $0 < \sigma < \min\{1, \frac{n}{p}\}$. Let Ω be a cone in $\mathbb{R}^n$ such that*

$$\begin{array}{lll} \Omega \ne \mathbb{R}^n & & \text{if } p > n; \\ \Omega \ne \mathbb{R}^n, & \Omega \ne \mathbb{R}^n \setminus \{0\} & \text{if } p = n. \end{array} \tag{10}$$

Then the infimum in (I_σ) is attained.

For $n \ge 3$ and $p = 2$ this result was obtained in [18].

Next Theorem provides the sharp constants in the Hardy inequality in cones.

Theorem 18 *Let $1 < p < \infty$, $\sigma = 0$. Let Ω be a cone in $\mathbb{R}^n$ satisfying (10). Then the infimum in (I_0) is not attained and equals $(\Lambda^{(p)}(G))^{\frac{1}{p}}$, where $G = \Omega \cap \mathbb{S}^{n-1}$ while*

$$\Lambda^{(p)}(G) = \min_{v \in \overset{o}{W}{}_p^1(G)\setminus\{0\}} \frac{\displaystyle\int_G \left(\left(\frac{n-p}{p}\right)^2 v^2 + |\nabla' v|^2\right)^{\frac{p}{2}} \mathrm{d}S}{\displaystyle\int_G |v|^p \, \mathrm{d}S} \tag{11}$$

(*here ∇' stands for the tangential gradient on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$*).

Proof First, the minimum in (11) is attained due to the compactness of embedding $\overset{o}{W}{}_p^1(G) \hookrightarrow L_p(G)$. Denote by V the minimizer of (11) normalized in $L_p(G)$. By standard argument, V is positive in G.

Let us define $U(r,\Theta)=r^{1-\frac{n}{p}}V(\Theta)$, where (r,Θ) are spherical coordinates in Ω. Then for any $h\in\mathcal{C}_0^1(\Omega)$ we have

$$\begin{aligned}
\int\limits_{\Omega}|\nabla U|^{p-2}\nabla U\cdot\nabla h\,\mathrm{d}x &= \int\limits_{\Omega}|\nabla U|^{p-2}(U_r h_r+\frac{1}{r^2}\nabla' U\cdot\nabla' h)\,\mathrm{d}x\\
&= \int\limits_{G}\Big(\big(\tfrac{n-p}{p}\big)^2V^2+|\nabla' V|^2\Big)^{\frac{p-2}{2}}V\cdot\int\limits_0^{\infty}\frac{p-n}{p}\,r^{\frac{n}{p}-1}h_r\,\mathrm{d}r\mathrm{d}S\\
&\quad+\int\limits_0^{\infty}\int\limits_{G}\Big(\big(\tfrac{n-p}{p}\big)^2V^2+|\nabla' V|^2\Big)^{\frac{p-2}{2}}\nabla' V\cdot\nabla' h\,r^{\frac{n}{p}-2}\,\mathrm{d}S\mathrm{d}r\\
&= \int\limits_0^{\infty}\int\limits_{G}\Big(\big(\tfrac{n-p}{p}\big)^2V^2+|\nabla' V|^2\Big)^{\frac{p-2}{2}}\Big(\big(\tfrac{n-p}{p}\big)^2Vh\\
&\quad+\nabla' V\cdot\nabla' h\Big)r^{\frac{n}{p}-2}\,\mathrm{d}S\mathrm{d}r\\
&\stackrel{*}{=}\Lambda^{(p)}(G)\int\limits_0^{\infty}\int\limits_{G}V^{p-1}h\,r^{\frac{n}{p}-2}\,\mathrm{d}S\mathrm{d}r=\Lambda^{(p)}(G)\int\limits_{\Omega}\frac{U^{p-1}}{r^p}h\,\,\mathrm{d}x
\end{aligned}\tag{12}$$

(the equality $(*)$ follows from the weak Euler–Lagrange equation for V). Thus, U is a positive weak solution of the equation $-\Delta_p u=\Lambda^{(p)}(G)\,\frac{u^{p-1}}{r^p}$.

The relation $\Lambda^{(p)}(G)\le\lambda_1^p(p,0,\Omega)$ follows now from [34, Theorem 2.3]. For the reader's convenience we reproduce the proof based on the so-called generalized Picone identity (see [2]).

For any $u\in\mathcal{C}_0^{\infty}(\Omega)$ we set $h=\frac{|u|^p}{U^{p-1}}$. Then (12) implies

$$\begin{aligned}
\Lambda^{(p)}(G)\int\limits_{\Omega}\frac{|u|^p}{r^p}\,\mathrm{d}x &= \Lambda^{(p)}(G)\int\limits_{\Omega}\frac{U^{p-1}}{r^p}h\,\mathrm{d}x=\int\limits_{\Omega}|\nabla U|^{p-2}\nabla U\cdot\nabla h\,\mathrm{d}x\\
&= \int\limits_{\Omega}\Big(p|\nabla U|^{p-2}\nabla U\cdot\nabla u\frac{|u|^{p-2}u}{U^{p-1}}-(p-1)|\nabla U|^p\frac{|u|^p}{U^p}\Big)\,\mathrm{d}x\\
&\stackrel{*}{\le}\int\limits_{\Omega}\Big(p|\nabla u|\cdot|\nabla U|^{p-1}\frac{|u|^{p-1}}{U^{p-1}}-(p-1)|\nabla U|^p\frac{|u|^p}{U^p}\Big)\mathrm{d}x\\
&\le\int\limits_{\Omega}|\nabla u|^p\,\mathrm{d}x.
\end{aligned}\tag{13}$$

Here $(*)$ is the Cauchy inequality while the last inequality follows from

$$x^p-pxy^{p-1}+(p-1)y^p\ge 0,\qquad x,y>0.\tag{14}$$

By approximation, (13) holds true for $u\in\dot{W}_p^1(\Omega)$.

To prove $\Lambda^{(p)}(G) = \lambda_1^p(p, 0, \Omega)$ we consider the sequence

$$u_\delta(r,\Theta) = \begin{cases} r^{1-\frac{n}{p}+\delta} V(\Theta), & r \le 1; \\ r^{1-\frac{n}{p}-\delta} V(\Theta), & r \ge 1. \end{cases}$$

Clearly, $u_\delta \in \dot{W}^1_p(\Omega)$. Direct computation shows

$$\begin{aligned}\int\limits_\Omega \left(|\nabla u_\delta|^p - \Lambda^{(p)}(G)\frac{|u_\delta|^p}{r^p}\right) \mathrm{d}x &= \frac{1}{p\delta}\int\limits_G \Bigg[\left(\left(\tfrac{n-p}{p}+\delta\right)^2 V^2 + |\nabla' V|^2\right)^{\frac{p}{2}} \\ &\quad + \left(\left(\tfrac{n-p}{p}-\delta\right)^2 V^2 + |\nabla' V|^2\right)^{\frac{p}{2}} - 2\Lambda^{(p)}(G) V^p\Bigg] \mathrm{d}S \\ &= \frac{1}{p\delta}\int\limits_G \Bigg[\left(\left(\tfrac{n-p}{p}+\delta\right)^2 V^2 + |\nabla' V|^2\right)^{\frac{p}{2}} \\ &\quad + \left(\left(\tfrac{n-p}{p}-\delta\right)^2 V^2 + |\nabla' V|^2\right)^{\frac{p}{2}} \\ &\quad - 2\left(\left(\tfrac{n-p}{p}\right)^2 V^2 + |\nabla' V|^2\right)^{\frac{p}{2}}\Bigg] \mathrm{d}S = O(\delta), \end{aligned} \tag{15}$$

and the statement follows.

Finally, the equality sign in $(*)$ means $\nabla u \parallel \nabla U$ while the equality in (14) means $x = y$. These two facts imply

$$\frac{\nabla u}{u} = \frac{\nabla U}{U} \qquad \Longrightarrow \qquad u = cU$$

on the set $\{u \neq 0\}$ and, therefore, in the whole Ω. Since $U \notin \dot{W}^1_p(\Omega)$, the equality in (13) is impossible. □

For $p < n$, $\Omega = \mathbb{R}^n$ and for $p > n$, $\Omega = \mathbb{R}^n \setminus \{0\}$ this result is well known, see, e.g., [24, Theorem 330]. For arbitrary cone the cases $p = 2$ and $p = n$ were considered in [32, Propositions 4.1 and 4.2].

Consider now the case $p > n$, $\Omega = \mathbb{R}^n \setminus \{0\}$, $\sigma > 0$. If the minimizer in (I_σ) was a radial function, the quantity $\lambda_1(p, \sigma, \Omega)$ could be expressed explicitely, see [11]. However, in general it is not the case.

Theorem 19 ([32, Theorem 4.1]). *Let $p > n$, $\Omega = \mathbb{R}^n \setminus \{0\}$. There exists $\widehat{\sigma}(n, p) < \frac{n}{p}$ such that for $\widehat{\sigma} < \sigma < \frac{n}{p}$ the minimizer in (I_σ) is a nonradial function.*

Corollary 1 *Let $p > n$, $\Omega = \mathbb{R}^n \setminus \{0\}$. Then for $\widehat{\sigma} < \sigma < \frac{n}{p}$ the problem (9) has at least two different positive solutions.*

Finally, we consider Ω being a perturbed cone.

Theorem 20 *Suppose that $1 < p < \infty$, $0 \le \sigma < \min\{1, \frac{n}{p}\}$, Ω_1 is a cone satisfying (10), $\Omega_2 \Subset \mathbb{R}^n \setminus \{0\}$ and $\Omega_1 \cap \Omega_2 \neq \emptyset$.*

1. *For $\Omega = \Omega_1 \setminus \overline{\Omega_2}$ the infimum in (I_σ) is not attained.*
2. *For $\Omega = \Omega_1 \cup \Omega_2$ the infimum in (I_σ) is attained if $\dot{W}^1_p(\Omega) \neq \dot{W}^1_p(\Omega_1)$.*

Proof **1**. For any $u \in \mathcal{C}_0^\infty(\Omega_1)$ there exists a dilation Π such that $\Pi u \in \mathcal{C}_0^\infty(\Omega)$. Due to the dilation invariance of (I_σ) we conclude that $\lambda_1(p, \sigma, \Omega) = \lambda_1(p, \sigma, \Omega_1)$.

Thus, if u minimizes the quotient (I_σ) on $\dot{W}_p^1(\Omega)$ then its zero continuation minimizes (I_σ) on $\dot{W}_p^1(\Omega_1)$. Therefore, it is the nonnegative solution of the problem (9). By Harnack's inequality ([39]), it is positive in Ω_1, a contradiction.

2. Let $\sigma > 0$. Then, by Theorem 17, there exists a function u positive in Ω_1 that minimizes the quotient (I_σ) on $\dot{W}_p^1(\Omega_1)$. If $\lambda_1(p, \sigma, \Omega) = \lambda_1(p, \sigma, \Omega_1)$ then the zero continuation of u minimizes (I_σ) on $\dot{W}_p^1(\Omega)$ that again leads to contradiction. Therefore, $\lambda_1(p, \sigma, \Omega) < \lambda_1(p, \sigma, \Omega_1)$.

Now let $\sigma = 0$. As in Theorem 18, we define a positive weak solution U of the equation $-\Delta_p u = \lambda_1^p(p, 0, \Omega_1)\,\frac{u^{p-1}}{r^p}$ in Ω_1. Then the relation (15) shows that U is the **ground state** for the functional

$$Q(u) = \int \left(|\nabla u|^p - \lambda_1^p(p, 0, \Omega_1)\frac{|u|^p}{r^p}\right) \mathrm{d}x,$$

that means Q is **degenerately positive** in Ω_1 ([34]; see also [35] for $p = 2$). By [34, Proposition 4.2], Q is **nonpositive** in Ω, i.e. $\lambda_1(p, 0, \Omega) < \lambda_1(p, 0, \Omega_1)$.

In both cases, the statement follows by the concentration-compactness principle. □

For $p = 2$ and $\sigma = 0$ this statement was proved in [36].

7 The Dirichlet problem with boundary singularity in bounded domains

In this section we treat the case $p = 2, 0 < \sigma < 1$. For the sake of brevity we use the notation $q = 2_\sigma^*$.

Let us denote by ϕ the extremal function of the problem (I_σ) for $p = 2$ and $\Omega = \mathbb{R}_+^n$. If we normalize ϕ by $\|\phi\|_{q,\sigma,\mathbb{R}_+^n} = 1$ then ϕ is a solution of the Dirichlet problem

$$-\Delta u = \lambda_1^2(2, \sigma, \mathbb{R}_+^n) \cdot \frac{u^{q-1}}{|x|^{q(1-\sigma)}} \quad \text{in} \quad \mathbb{R}_+^n, \qquad u\big|_{x_n=0} = 0.$$

By standard elliptic theory, see, e.g., [27], $\phi \in \mathcal{C}^2(\overline{\mathbb{R}_+^n} \setminus \{0\})$.

Theorem 21 ([16, Theorem 2.1]). *Let $n \geq 2$. Then, for small $|x|$*

$$\phi(x) \sim C x_n, \qquad |\nabla\phi(x)| \sim C;$$

for large $|x|$

$$\phi(x) \sim \frac{C x_n}{|x|^n}, \qquad |\nabla\phi(x)| \asymp \frac{C}{|x|^n}.$$

Now let Ω be a bounded domain, and let $\partial\Omega \in \mathcal{C}^1$ in some neighborhood of the origin. We define the Cartesian coordinates y with $y' = (y_1, \ldots, y_{n-1})$ in the tangent plane and Oy_n directed into Ω. Then, in a neighborhood of the origin, $\partial\Omega$ is given by the equation $y_n = F(y')$. It is easily seen that $F \in \mathcal{C}^1$ and $F(y') = o(|y'|)$.

We assume that $\partial\Omega$ is **average concave** in a neighborhood of the origin, i.e. for sufficiently small r

$$f(r) := \fint_{r\mathbb{S}^{n-2}} F(y') < 0$$

(here the dashed integral stands for the mean value). Trivially $f \in \mathcal{C}^1$.

We assume also that f is **regularly varying** at zero i.e., given $t > 0$,

$$\lim_{r\to 0} \frac{f(tr)}{f(r)} = t^\gamma, \tag{16}$$

and $1 \le \gamma \le n+1$.

It is well known, see, e.g. [37], that (16) implies $f(r) = -r^\gamma \psi(r)$ where ψ is a slowly varying function at zero. Note that for $\gamma = 1$ the relation $F(y') = o(|y'|)$ implies $\lim\limits_{r\to 0} \psi(r) = 0$. For $\gamma = n+1$ we assume in addition that $\int_0^1 \frac{\psi(s)}{s}\,ds = +\infty$.

Let us introduce the function

$$f_2(r) := \fint_{r\mathbb{S}^{n-2}} |\nabla F(y')|^2.$$

Assume that

$$\lim_{r\to 0} \frac{f_2(r)}{f(r)}\, r = 0. \tag{17}$$

Theorem 22 ([16, Theorem 3.1]). *Let $n \ge 2$. Let $\partial\Omega$ satisfy the above assumptions. Then for $p = 2$ and for arbitrary $0 < \sigma < 1$ the infimum in (I_σ) is attained.*

In [20] this result was obtained for $n \ge 4$ and $\partial\Omega \in \mathcal{C}^2$, all principal curvatures of $\partial\Omega$ at the origin were assumed negative. Some calculation errors in [20] do not influence on the result. In [21,22] the statement of Theorem 22 was proved under the assumption that the mean curvature of $\partial\Omega$ at the origin is negative. [21] also deals only with $n \ge 4$, in [22] the case $n \ge 3$ is considered.

Note that if $H(0) < 0$ then, as $r \to 0$,

$$f(r) \sim -Cr^2, \qquad f_2(r) \sim Cr^2.$$

Thus, the relation (16) holds true with $\gamma = 2$. The relation (17) is also fulfilled. We underline that the hypotheses of Theorem 22 do not assume even the existence of the mean curvature (if $\gamma < 2$). On the other hand, for $\gamma > 2$, $H(0) = 0$. Moreover, the assumption (17) can satisfy even if the main term of the asymptotic expansion of F vanishes under average. For example, it is the case if $F(y') = y_1^3 - y_2^4$.

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